De Morgan's law: $(A\cap B)^c = A^c \cup B^c$ $\quad (\bigcap_{i\in I} A_i)^c = \bigcup_{i\in I} A_i^c$

$(A\cup B)^c = A^c \cap B^c$. $\quad (\bigcup_{i\in I} A_i)^c = \bigcap_{i\in I} A_i^c$

$$\mathbb{R}\setminus \bigcap_{n=1}^{\infty}\left(-\tfrac{1}{n},\tfrac{1}{n}\right) = \left[\bigcap_{n=1}^{\infty}\left(-\tfrac{1}{n},\tfrac{1}{n}\right)\right]^c$$

$$= \bigcup_{i=1}^{\infty}\left(\left(-\infty,-\tfrac{1}{n}\right]\cup\left[\tfrac{1}{n},+\infty\right)\right)$$

$$= (-\infty,0)\cup(0,+\infty) = \mathbb{R}\setminus\{0\}.$$

Examples of limits: $\lim_{n\to\infty}\frac{1}{n} \overset{①}{=} 0$

Using this limit, together with some other theorems, you can compute, e.g.

$$\lim_{n\to\infty}\frac{2n+1}{5n+4} = \lim_{n\to\infty}\frac{2+\frac{1}{n}}{5+\frac{4}{n}} \overset{②}{=} \frac{2+\lim_{n\to\infty}\frac{1}{n}}{5+\lim_{n\to\infty}\frac{1}{n}} = \frac{2}{5}.$$

Issues: Why is ①, ② really true.

This is one of the main focus of 311 class.

Recall: Sequence: a function $\mathbb{Z}_+ \longrightarrow \mathbb{R}$.

usually denoted $(a_n)_{n=1}^{\infty}$. (or $\{a_n\}$, (a_n), $\cdots$)

Convergent sequence: a sequence satisfying, $\exists\, a\in\mathbb{R}$, s.t.

$$\forall \varepsilon>0,\ \exists N>0,\ \forall n>N,\ |a_n-a|<\varepsilon.$$

If this is true, a is called the limit of the sequence.

Motivation: Consider $a_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots+(-1)^{n-1}\frac{1}{n}$. (partial sum)

$\lim_{n\to\infty} a_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+(-1)^{n-1}\frac{1}{n}+\cdots$ (infinite series).

Calc II. Alternating Series test:

if $a_n \searrow 0$, then series $\sum_{n=1}^{\infty}(-1)^n a_n$ converges

$\Rightarrow \lim_{n\to\infty} a_n$ exists.

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \cdots$$

$$\frac{1}{2}S = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12}$$

$$\frac{3}{2}S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$

RHS differs to S just by a rearrangement (however infinite).

But the limit changes to $\frac{3}{2}S$.

In fact, not all the usual operation apply to infinite case.

It's one of the central topics in this class to determine if certain inf. operation applies.

In order to achieve that, you have to start from the basic & annoying definition of limits, to see why usual operations work, when it doesn't work, etc..

<u>History</u>: Idea of limit starts for Archimede.

inherited by Newton to invent calculus.

Within 200 years after Newton, people used calculus to get correct / absurd results. e.g. $1-1+1-1+1-1+1-1+\cdots = \sum_{n=1}^{\infty}(-1)^{n-1}$

$$S_n = \sum_{i=1}^{n}(-1)^{n-1} \Rightarrow S_n = \begin{cases} 1 & n \text{ is odd.} \\ 0 & n \text{ is even.} \end{cases}$$

Euler believed $1-1+1-1+1-1+\cdots = \frac{1}{2}$ (absurd but makes sense again after complex functions)

Until Cauchy in 1840's who developed the rigorous definition of limits, people finally know how to tell if certain computation makes sense.

<u>Basic idea of ε-N language:</u> recognize infinity not as something static, but a dynamic process!

<u>Example:</u> $\lim_{n\to\infty} \frac{1}{n} = 0.$

$a_n = \frac{1}{n}, \quad a_1 = 1, \; a_2 = \frac{1}{2}, \; a_3 = \frac{1}{3}, \; a_4 = \frac{1}{4}, \; \cdots, \; a_n = \frac{1}{n}, \cdots$

NONE of these ACTUALLY takes ZERO.

But they DO get closer and closer to ZERO

AS n BECOMES LARGER and LARGER.

Using Cauchy's language, $\lim_{n\to\infty} \frac{1}{n} = 0$ means

$\forall \varepsilon > 0, \underline{\forall n >> 0}. \; |a_n - 0| < \varepsilon.$ $\quad$ a_n is close enough to 0.

$\downarrow$

$\exists N > 0, \; \forall n > N.$ $\quad$ n large enough

To prove that, we have to use Archimedean's property

$\forall x \in \mathbb{R}, \exists n \in \mathbb{N}, \text{ s.t. } n > x$

$(\forall x > 0, \; \exists n \in \mathbb{N}, \text{ s.t. } \frac{1}{n} < x)$

$\forall \varepsilon > 0,$ Want $|a_n| = \frac{1}{n} < \varepsilon$

Pick $N \in \mathbb{N}$, s.t. $N > \frac{1}{\varepsilon}$ (or $\frac{1}{N} < \varepsilon$)

Then $\forall n > N, \; |a_n - 0| = \frac{1}{n} < \frac{1}{N} < \varepsilon$

Note: ε must be arbitrary.

Choice of N must allow ε to vary arbitrarily

(choice must be depedent to ε).

The lower n must be arbitrarily longer than N.

$\varepsilon = 0.1$, Pick $N = 11$, $\Rightarrow \left|\frac{1}{11} - 0\right| < \frac{1}{10} = \varepsilon$

$\forall n > N$, $a_n = \frac{1}{n} < \frac{1}{11} < \varepsilon$.

$\varepsilon = 0.01$, Pick $N = 101 \Rightarrow \left|\frac{1}{101} - 0\right| < \frac{1}{100} = \varepsilon$

$\forall n > N$, $a_n = \frac{1}{n} < \frac{1}{101} < \varepsilon$.

As ε changes, the choice of N changes accordingly in general.

(dynamic process)

Example: Prove that $\lim_{n\to\infty} \frac{2n+1}{5n+4} = \frac{2}{5}$ w/o. using ①. ②

Proof: $\forall \varepsilon > 0$, want to find N, s.t. $\forall n > N$, $\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| < \varepsilon$.

$$\frac{2n+1}{5n+4} - \frac{2}{5} = \frac{10n+5-(10n+8)}{5(5n+4)} = \frac{-3}{5(5n+4)}$$

If $b > 0$, then $|a| < b \Leftrightarrow -b < a < b$.

Want $N \in \mathbb{N}$, $-\varepsilon < \frac{-3}{5(5n+4)} < \varepsilon \quad \forall n > N$

guaranteed for free.

$$\varepsilon > \frac{3}{5(5n+4)} \Rightarrow 25n + 20 > \frac{3}{\varepsilon}$$

$$\Rightarrow n > \frac{3}{25\varepsilon} - \frac{4}{5}.$$

Pick $N > \frac{3}{25\varepsilon} - \frac{4}{5}$ any integer, so $\forall n > N$.

$$\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| = \frac{3}{5(5n+4)} < \frac{3}{5(5N+4)} < \varepsilon.$$

Example: $a_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$

Show that $\lim_{n\to\infty} a_n \neq 1$ or 0.

$|a_n - 1|$ is either 0 or 1.

Recall: $\neg(\forall \varepsilon > 0, \exists N > 0, \forall n > N, |a_n - a| < \varepsilon)$

$= (\exists \varepsilon > 0, \forall N > 0, \exists n > N, |a_n - a| \geq \varepsilon)$

Pick $\varepsilon = \frac{1}{2}$, $\forall N > 0$, $\exists n > N$, (pick any even n)

$|a_n - 1| = |0 - 1| = 1 \geq \frac{1}{2}$.

$\Rightarrow 1$ is not $\lim_{n\to\infty} a_n$.

(Similarly, $0, \frac{1}{2} \neq \lim_{n\to\infty} a_n$)

But $a_n = 1$ for infinitely many n's.

$\forall \varepsilon > 0$, $\exists N > 0$ $\exists$ inf. many $n > N$, $|a_n - 1| = 0 < \varepsilon$.

however we need $\forall n > N$.

Exercise: Show that $\frac{1}{2} \neq \lim_{n\to\infty} a_n$ (so Euler was wrong).